Teorema lui Cauchy

1. enuntul teoremei
2. demonstratia teoremei
3. interpretare geometrica
4. aplicatii

1. ENUNTUL TEOREMEI

Fie f si g doua functii, f,g:[a,b]→R, cu proprietatile:

a) f si g continue pe [a,b]
b) f si g derivabile pe (a,b)
c) g'(x)=0

atunci g(a)=g(b) si (∃) cel putin un punct c∈(a,b) a.i.

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}$$

2. DEMONSTRATIA TEOREMEI

$$\left.\begin{array}{l}\textbf{P.p.a. ca } \mathbf{g(a)=g(b)}\\ \mathbf{g}\ \textbf{ continua pe } \mathbf{[a,b]}\\ \mathbf{g}\ \textbf{ der. pe } \mathbf{(a,b)}\end{array}\right\}\Rightarrow(\exists)\ \textbf{cel putin un punct } \mathbf{c\in(a,b)}\ \textbf{a.i. } \mathbf{g'(c)=0}$$

$\textbf{dar } \mathbf{g'(x)\neq 0 \Rightarrow}$ **contradictie.**

$\Rightarrow \mathbf{g(a)\neq g(b).}$

$\textbf{Fie } \mathbf{h(x)=f(x)+k\cdot g(x); k}\ \textbf{- constanta reala a.i. } \mathbf{h(a)=h(b)}$

$\Rightarrow \mathbf{f(a)+k\cdot g(a)=f(b)+k\cdot g(b)}$

$\Rightarrow \mathbf{f(a)-f(b)=k\cdot g(b)-k\cdot g(a)}$

$\Rightarrow \mathbf{f(a)-f(b)=k(g(b)-g(a))}$

$$\Rightarrow \mathbf{k=-\frac{f(b)-f(a)}{g(b)-g(a)}}$$

$$\left.\begin{array}{l}\mathbf{h\ \ continua\ pe\ [a,b]}\\ \mathbf{h\ \ derivabila\ pe\ (a,b)}\\ \mathbf{h(a)=h(b)}\end{array}\right\}\Rightarrow(\exists)\ \mathbf{cel\ putin\ un\ punct\ c}\in\mathbf{(a,b)\ a.i.\ h'(c)=0;}$$

$$\mathbf{h'(x)=f'(x)+k\cdot g'(x)}$$

$$\mathbf{h'(c)=f'(c)+k\cdot g'(c)}$$

$$\Rightarrow\mathbf{f'(c)+k\cdot g'(c)=0}$$

$$\Rightarrow\mathbf{k=-\frac{f'(c)}{g'(c)}}$$

$$\Rightarrow\mathbf{\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}}$$

3.INTERPRETARE GEOMETRICA

Pantele celor doua drepte sunt proportionale cu pantele tangentelor duse la graficul functiei in punctul c corespunzator.

4. APLICATII

1. Sa se aplice TEOREMA LUI CAUCHY in cazul functiilor :

$f : [-2,5] \to R$

$$f(x) = \begin{cases} \sqrt{x+3} & ; -2 \le x < 1 \\ \frac{x}{4} + \frac{7}{4} & ; 1 \le x \le 5 \end{cases}$$

$g : [-2,5] \to R \quad g(x) = x.$

Functia f este continua pe [-2,1) si [1,5] ca functie elementara.

Se pune problema in $x = 1$

$$\lim_{x \to 1; x<1} f(x) = \lim_{x \to 1; x<1} \sqrt{x+3} = 2$$

$$\lim_{x \to 1; x>1} f(x) = \lim_{x \to 1; x>1} \frac{x+7}{4} = 2$$

$f(1) = 2 \Rightarrow f$ este continua pe $[-2,5]$

Functia f este derivabila pe [-2,1) si [1,5] ca functie elementara .

Se pune problema in $x = 1$

$$f'_S(1) = \lim_{x \to 1; x<1} \frac{f(x) - f(1)}{x - 1} = \frac{1}{4}$$

$$f'_D(1) = \lim_{x \to 1; x<1} \frac{f(x) - f(1)}{x - 1} = \frac{1}{4}$$

$\Rightarrow$ f este derivabila pe[-2,5].

Functia g este continua si derivabila pe [-2,5] ca f. elementara.

$$g'(x) = 1 \neq 0$$

$$\xrightarrow{t.cauchy} (\exists) \text{cel putin un } c \in (-2,5) \text{ a.i. } \frac{f'(c)}{g'(c)} = \frac{f(5)-f(-2)}{g(5)-g(-2)}$$

$$f'(x) = \begin{cases} \dfrac{1}{2\sqrt{x-3}} & ; x \in (-2,1) \\ \dfrac{1}{4} & ; x \in [1,5) \end{cases}$$

$$g'(x) = 1$$

$$\frac{f'(c)}{g'(c)} = \frac{2}{7}$$

Cazul 1 : $x \in (-2,1)$

$$\Rightarrow f'(c) = \frac{1}{2\sqrt{c-3}} \; ; \; g'(c) = 1 \Rightarrow \frac{1}{2\sqrt{c-3}} = \frac{2}{7}$$

$$4\sqrt{c-3} = 7 \Rightarrow 16(c-3) = 49 \Rightarrow c = \frac{97}{16} \in (-2,1)$$

Cazul 2 : $x \in (1,5)$

$$\Rightarrow f'(c) = \frac{1}{4} \; ; g'(c) = 1 \Rightarrow \frac{1}{4} = \frac{2}{7} \text{ (F)}$$

$$\Rightarrow c = \frac{97}{16}$$

2. Sa se aplice Teorema lui Cauchy pentru functiile $f, g : [1, e] \to R$, $f(x) = \ln(x)$ si $g(x) = 2x - 1$, determinand punctul c corespunzator. Similar pentru functiile $f, g : \left[\frac{\pi}{6}, \frac{\pi}{3}\right] \to R$ $f(x) = \sin(x)$ si $g(x) = \cos(x)$.

$\bullet\, f : [1, e] \to R \quad f(x) = \ln(x)$

$g : [1, e] \to R \quad g(x) = 2x - 1$

f si g sunt continue pe [1,e] ca functii elementare

f si g sunt derivabile pe (1,e) ca functii elementare

$g'(x) = (2x - 1)' = 2$

$\xrightarrow[\Rightarrow]{\text{t. cauchy}}$ $(\exists)$ cel putin un punct $c \in (1, e)$ a.i. $\frac{f'(c)}{g'(c)} = \frac{f(e) - f(1)}{g(e) - g(1)} = \frac{1}{2e - 2}$

$f'(x) = \frac{1}{x}$; $g'(x) = 2$

$$\frac{f'(c)}{g'(c)} = \frac{1}{2c} = \frac{1}{2e - 2} \Rightarrow 2c = 2(e - 1) \Rightarrow c = e - 1.$$

$\bullet\, f : \left[\frac{\pi}{6}; \frac{\pi}{3}\right] \to R \quad f(x) = \sin(x)$

$g : \left[\frac{\pi}{6}; \frac{\pi}{3}\right] \to R \quad g(x) = \cos(x)$

$$\text{f si g sunt continue pe} \left[\frac{\pi}{6};\frac{\pi}{3}\right] \text{ca f elementare}$$

$$\text{f si g sunt der. pe} \left[\frac{\pi}{6};\frac{\pi}{3}\right] \text{ca f elementare}$$

$$g'(x) = -\sin(x); \sin(x) \neq 0 \Rightarrow x \neq k\pi$$

$$\xrightarrow[\Rightarrow]{\text{t. cauchy}} (\exists) \text{ cel putin un punct } c \in \left(\frac{\pi}{6};\frac{\pi}{3}\right)$$

$$\text{a.i. } \frac{f'(c)}{g'(c)} = \frac{f\left(\frac{\pi}{3}\right) - f\left(\frac{\pi}{6}\right)}{g\left(\frac{\pi}{3}\right) - g\left(\frac{\pi}{6}\right)} = -1$$

$$\frac{f'(c)}{g'(c)} = \frac{\cos(c)}{-\sin(c)} = -ctg(c) = -1 \Rightarrow c = \frac{\pi}{4}$$

3. Sa se studieze valabilitatea teremei lui Cauchy si sa se determine valoarea punctului c :

$$f:[0,3] \to R \quad f(x) = \begin{cases} \frac{x^3}{3} - x^2 + 1, & x \in (1,3] \\ -x + \frac{4}{3}, & x \in [0,1] \end{cases}$$

$$g:[0,3] \to R \quad g(x) = x$$

Functia f este continua si derivabila pe [0,1) si (1,3]
ca functie elementara.
Se pune problema in $x = 1$.
Continuitatea

$$\lim_{x\to 1; x<1} f(x) = \lim_{x\to 1; x<1} -x + \frac{4}{3} = 1/3$$

$$\lim_{x\to 1; x>1} f(x) = \lim_{x\to 1; x>1} \frac{x^3}{3} - x^2 + 1 = 1/3$$

$f(1) = 1/3 \Rightarrow f$ este continua pe [0,3].
Derivabilitatea

$$f'_S(1) = \lim_{x\to 1; x<1} \frac{f(x) - f(1)}{x - 1} = -1$$

$$f'_D(1) = \lim_{x\to 1; x>1} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1; x>1} \frac{x^3 - 3x^2 + 2}{3(x-1)} = -1$$

$\Rightarrow f$ este der. pe (0,3).
Functia g este continua si derivabila pe [0,3]
ca functie elementara.

$$g'(x) = 1 \xrightarrow{\text{t. cauchy}} (\exists) \text{ cel putin un punct } c \in (0,3)$$

$$\text{a.i. } \frac{f'(c)}{g'(c)} = \frac{f(3) - f(0)}{g(3) - g(0)} = -1/9$$

$$f'(x)=\begin{cases} x^2-2x & x\in(1,3) \\ -1 & x\in(0,1) \end{cases}$$

$$g'(x)=1$$

Cazul 1 : $x\in(1,3)$

$$f'(c)=c^2-2c\ ;\ g'(c)=1 \Rightarrow \frac{f'(c)}{g'(c)}=c^2-2c=-1/9$$

$$\Rightarrow 9c^2-18c+1=0$$

$$\Delta=288\ \Rightarrow c_1=\frac{3+2\sqrt{2}}{3}\in(1,3)$$

Cazul 2 : $x\in(0,1)$

$$f'(c)=-1\quad g'(c)=1\ \Rightarrow -1=-1\,(A)$$

$$\Rightarrow c=\frac{3+2\sqrt{2}}{3}$$