Probleme rezolvate

1) Să se calculeze limitele următoarelor şiruri:

a) $x_n = \frac{1}{n}\sqrt[n]{(n+1)(n+2)...(n+n)}, n \geq 2$

b) $x_n = \frac{1}{n^2}\left(3e^{\frac{\sqrt{1\cdot 2}}{n}} + 5e^{\frac{\sqrt{2\cdot 3}}{n}} + ... + (2n-1)e^{\frac{\sqrt{(n-1)\cdot n}}{n}}\right)$

Soluție

a) $x_n = \sqrt[n]{\frac{(n+1)(n+2)...(n+n)}{n^n}} = \sqrt[n]{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)...\left(1+\frac{n}{n}\right)}$.

$\ln x_n = \frac{1}{n}\sum_{k=1}^{n}\ln\left(1+\frac{k}{n}\right)$. Folosind consecința 5.1.2 avem:

$$\lim_{n\to\infty}\ln x_n = \int_0^1 \ln(1+x)dx = x\ln(1+x)\Big|_0^1 - \int_0^1 \frac{x}{1+x}dx =$$

$$\ln 2 - \left(x - \ln(1+x)\right)\Big|_0^1 = 2\ln 2 - 1 = \ln\frac{4}{e} \Rightarrow \lim_{n\to\infty} x_n = \frac{4}{e}$$

b) Considerăm $f, g : [0,1] \to R$, $f(x) = x, g(x) = e^x$.

Alegem $\Delta_n = \left(0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n}{n}\right)$, $\|\Delta_n\| \to 0$; $x_i^n = \frac{i}{n}$

$\alpha_i^n = f\left(\frac{x_{i-1}^n + x_i^n}{2}\right)$, $\beta_i^n = g\left(\sqrt{x_{i-1}^n \cdot x_i^n}\right)$, $i = \overline{1,n}$

$$x_n = \frac{1}{n}\sum_{i=1}^{n}\frac{2i-1}{n}e^{\frac{\sqrt{i(i-1)}}{n}} - \frac{1}{n^2} = \frac{2}{n}\sum_{i=1}^{n}\frac{x_{i-1}^n + x_i^n}{2}\cdot e^{\sqrt{x_{i-1}^n\cdot x_i^n}} - \frac{1}{n^2} =$$

$$2\sum_{i=1}^{n}\alpha_i^n\beta_i^n\left(x_i^n - x_{i-1}^n\right) - \frac{1}{n^2}.$$

Folosind Teorema 5.1.3, obținem:

$$\lim_{n\to\infty} x_n = 2\int_0^1 xe^x dx = 2\left(xe^x - e^x\right)\Big|_0^1 = 2$$

2) Să se arate că $\forall\, n \in N^*$, ecuația $\frac{1}{\sqrt{nx+1}} + \frac{1}{\sqrt{nx+2}} + ... + \frac{1}{\sqrt{nx+n}} = \sqrt{n}$ are o soluție unică $x = x_n$. Să se calculeze $\lim_{n\to\infty} x_n$

Soluție

Pentru orice $n \in N^*$, definim $f_n : [0,\infty) \to R$, $f_n(x) = \sum_{k=1}^{n}\frac{1}{\sqrt{x+\frac{k}{n}}}$. Ecuația din enunț se scrie $f_n(x) = n$. Pentru *n* fixat avem

$f_n(x)-n>1+1+...+1-n=n-n=0$ şi $\lim\limits_{n\to\infty} f_n(x)-n=-n<0$ deci $\exists\, x_n\in(0,\infty)$ astfel încât $f_n(0)=n$ (proprietatea lui Darboux). Soluția x_n este unică deoarece f_n este strict descrescătoare. Vom arăta că $\lim\limits_{n\to\infty} x_n=\dfrac{9}{16}$.

Obținerea limitei este sugerată de forma ecuației $\dfrac{1}{n}\sum\limits_{k=1}^{n}\dfrac{1}{\sqrt{x_n+\dfrac{k}{n}}}=1$ care conduce la ecuația în l următoare:

$\int_0^1\dfrac{1}{\sqrt{l+t}}dt=1$ cu soluție unică $l=\dfrac{9}{16}$.

Fie $\mu>0$. Pentru orice $n\in N^*$ notăm: $a_n=\dfrac{1}{n}\sum\limits_{k=1}^{n}\dfrac{1}{\sqrt{\dfrac{9}{16}+\mu+\dfrac{k}{n}}}$ şi

$b_n=\dfrac{1}{n}\sum\limits_{k=1}^{n}\dfrac{1}{\sqrt{\dfrac{9}{16}-\mu+\dfrac{k}{n}}}$ (se ia şi $\mu<\dfrac{9}{16}$). Atunci $\lim\limits_{n\to\infty} a_n=s<1$ şi $\lim\limits_{n\to\infty} b_n=t>1$

$$s=\int_0^1\frac{1}{\sqrt{\frac{9}{16}+\mu+t}}dt=2\left(\sqrt{\frac{25}{16}+\mu}-\sqrt{\frac{9}{16}+\mu}\right) \text{ şi}$$

$$t=\int_0^1\frac{1}{\sqrt{\frac{9}{16}-\mu+t}}dt=2\left(\sqrt{\frac{25}{16}-\mu}-\sqrt{\frac{9}{16}-\mu}\right).$$

$\exists\, n(\mu)\in N$ astfel încât $\forall\, n\ge n(\mu)$ să avem $a_n<1<b_n$, adică

$\dfrac{1}{n}f_n\left(\dfrac{9}{16}+\mu\right)<\dfrac{1}{n}f_n\left(x_n\right)<\dfrac{1}{n}f_n\left(\dfrac{9}{16}-\mu\right)$, deci $\dfrac{9}{16}-\mu<x_n<\dfrac{9}{16}+\mu$.

Cum μ este arbitrar (de mic), obținem $\lim\limits_{n\to\infty} x_n=\dfrac{9}{16}$.

3) Fie $f:[0,\infty)\to R$ o funcție periodică, de perioadă 1, integrabilă pe $[0,1]$. Pentru un şir $\left(x_n\right)_{n\ge 0}, x_0=0$, strict crescător şi nemărginit cu $\lim\limits_{n\to\infty}\left(x_{n+1}-x_n\right)=0$, notăm $r(n)=\max\left\{k \mid x_n\le n\right\}$.

a) Să se arate că: $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{r(n)}\left(x_k-x_{k-1}\right)f(x_k)=\int_0^1 f(x)dx$

b) Utilizând eventual rezultatul de la punctul a), demonstrați că:

$\lim\limits_{n\to\infty}\dfrac{1}{\ln n}\sum\limits_{k=1}^{n}\dfrac{f(\ln k)}{k}=\int_0^1 f(x)dx$

Soluție

a) Avem $a_n = \frac{1}{n}\sum_{p=1}^{n}\left(\sum_{p-1<x_k\le p}\left(x_{k+1}-x_k\right)f\left(x_k\right)\right) = \frac{1}{n}\sum_{p=1}^{n}s_p$.

Cu Cesaro-Stolz avem: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n$

$s_n = \sum_{n-1<x_k\le n}\left(x_{k+1}-x_k\right)f(x_k) = \sum_{0<x_k-(n-1)\le 1}\left(y_{k+1}-y_k\right)f(y_k)$, cu $y_k = x_k-(n-1)$,

reprezintă suma Riemann asociată funcției *f* şi diviziunii $(y_k)_{r(n-1)<k\le r(n)}$ a intervalului $[0,1]$, a cărei normă tinde la 0.

Prin urmare, $\lim_{n\to\infty} s_n = \int_0^1 f(x)dx$

b) Pentru $x_n = \ln n$, rezultă că $z_n = \frac{1}{n}\sum_{k=1}^{\left[e^n\right]}\ln\frac{k+1}{k}f(\ln k) \to I = \int_0^1 f(x)dx$.

Avem $\lim_{n\to\infty} s_{[\ln n]} = I$, deci $\frac{1}{\left[\ln n\right]}\sum_{k=1}^{\left[e^{[\ln n]}\right]}\ln\frac{k+1}{k}f(\ln k)\to I \Rightarrow$

$$\frac{1}{\ln n}\sum_{k=1}^{\left[e^{[\ln n]}\right]}\ln\frac{k+1}{k}f\left(\ln k\right)\to I$$

Apoi

$$\frac{1}{\ln n}\sum_{k=1}^{n}\ln\frac{k+1}{k}f\left(\ln k\right) = \frac{1}{\ln n}\sum_{k=1}^{\left[e^{[\ln n]}\right]}\ln\frac{k+1}{k}f\left(\ln k\right) + \frac{1}{\ln n}\sum_{k=\left[e^{[\ln n]}\right]+1}^{n}\ln\frac{k+1}{k}f\left(\ln k\right)$$

şi arătăm că $\lim_{n\to\infty}\sum_{k=\left[e^{[\ln n]}\right]+1}^{n}\ln\frac{k+1}{k}f\left(\ln k\right) = 0$

Cu $M = \sup\left|f(x)\right|$, avem:

$$\left|\sum_{k=\left[e^{[\ln n]}\right]+1}^{n}\ln\frac{k+1}{k}f(\ln k)\right| \le M\sum_{k=\left[e^{[\ln n]}\right]+1}^{n}\ln\frac{k+1}{k} = M\ln\frac{n}{\left[e^{[\ln n]}\right]+1}\to 0$$

Deci

$$\left|\frac{1}{\ln n}\sum_{k=1}^{n}\left(\frac{1}{k}-\ln\frac{k+1}{k}\right)f\left(\ln k\right)\right| \le M\frac{1}{\ln n}\sum_{k=1}^{n}\left(\frac{1}{k}-\ln\frac{k+1}{k}\right) = M\frac{1+\ldots+\frac{1}{n}-\ln\left(n+1\right)}{\ln n}\to 0$$

ANALIZĂ MATEMATICĂ clasa a XI-a

1.Limite de șiruri

Să se calculeze limitele:

1.	$\lim_{n\to\infty} n^3 = \infty$	$\lim_{n\to\infty}(-n^2+n-1)$
2.	$\lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty}\frac{1}{2\sqrt{n}} = 0$	$\lim_{n\to\infty}\frac{n}{\sqrt{4n^2+1}}$
3.	$\lim_{n\to\infty}\frac{(n-1)^3}{2n^3} = \frac{\infty}{\infty} = \lim_{n\to\infty}\frac{n^3}{2n^3} = \frac{1}{2}$	$\lim_{n\to\infty}\frac{n^2}{(1-2n)^2}$
4.	$\lim_{n\to\infty}\frac{1+2+\cdots+n}{n^2} = \frac{\infty}{\infty} = \lim_{n\to\infty}\frac{n(n+1)}{2n^2} = \frac{1}{2}$	$\lim_{n\to\infty}\frac{1^2+2^2+\cdots+n^2}{n^3}$
5.	$\lim_{n\to\infty}\left(\sqrt{n^2+5n+1}-n\right) = \infty-\infty$ $a-b=\frac{a^2-b^2}{a+b}$ $= \lim_{n\to\infty}\frac{n^2+5n+1-n^2}{\sqrt{n^2+5n+1}+n} = \lim_{n\to\infty}\frac{5n}{2n} = \frac{5}{2}$	$\lim_{n\to\infty}\left(n-\sqrt{n^2-4n+1}\right)$
6.	$\lim_{n\to\infty}\left(\sqrt{n^2+5n+1}-\sqrt{n^2-7n+3}\right) = \infty-\infty$ $= \lim_{n\to\infty}\frac{n^2+5n+1-n^2+7n-3}{n+n} = \lim_{n\to\infty}\frac{12n}{2n} = 6$	$\lim_{n\to\infty}\left(\sqrt{n^2-1}-\sqrt{n^2+9n}\right)$
7.	$\lim_{n\to\infty}\left(\sqrt[3]{n^3+5n^2+1}-n\right) = \infty-\infty$ $a-b=\frac{a^3-b^3}{a^2+ab+b^2}$ $= \lim_{n\to\infty}\frac{n^3+5n^2+1-n^3}{\sqrt[3]{(n^3+5n^2+1)^2}+n\sqrt[3]{n^3+5n^2+1}+n^2}$ $= \lim_{n\to\infty}\frac{5n^2+1}{n^2+n^2+n^2} = \lim_{n\to\infty}\frac{5n^2}{3n^2} = \frac{5}{3}$	$\lim_{n\to\infty}\left(n-\sqrt[3]{n^3-6n^2+1}\right)$
8.	$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = 1^\infty =$	$\lim_{n\to\infty}\left(\frac{n-1}{n}\right)^n$

	$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$ $$=\lim_{n\to\infty}\left(1+\frac{n}{n+1}-1\right)^n = \lim_{n\to\infty}\left(1+\frac{-1}{n+1}\right)^n =$$ $$=\lim_{n\to\infty}\left[\left(1+\frac{-1}{n+1}\right)^{\frac{n+1}{-1}}\right]^{\frac{-1}{n+1}n} = e^{\lim_{n\to\infty}\frac{-n}{n+1}} = e^{-1}$$	
9.	$$\lim_{n\to\infty}\left(\frac{n^2-1}{n^2+n+1}\right)^{\frac{(3n-1)^2}{n}} = 1^\infty =$$ $$=\lim_{n\to\infty}\left(1+\frac{n^2-1}{n^2+n+1}-1\right)^{\frac{(3n-1)^2}{n}}$$ $$=\lim_{n\to\infty}\left(1+\frac{-n-2}{n^2+n+1}\right)^{\frac{(3n-1)^2}{n}}$$ $$=\lim_{n\to\infty}\left[\left(1+\frac{-n-2}{n^2+n+1}\right)^{\frac{n^2+n+1}{-n-2}}\right]^{\frac{-n-2}{n^2+n+1}\cdot\frac{(3n-1)^2}{n}}$$ $$=e^{\lim_{n\to\infty}\frac{-n}{n^2}\cdot\frac{9n^2}{n}} = e^{-9}$$	$$\lim_{n\to\infty}\left(\frac{3n^2+1}{n^2+n+1}\right)^n$$
10.	$\lim_{n\to\infty}(1+2^n+3^n) = \lim_{n\to\infty}3^n = \infty, \qquad 3>1$ $$\lim_{n\to\infty}a^n = \begin{cases}\infty, a>1\\ 1, a=1\\ 0, \|a\|<1\\ \nexists, a\le -1\end{cases}$$	$$\lim_{n\to\infty}\left(1+\left(\frac{1}{2}\right)^n+\left(\frac{1}{3}\right)^n\right)$$
11.	$$\lim_{n\to\infty}\frac{3^n+2^n}{\left(\frac{2}{3}\right)^n+2^n+3^{n-1}} = \lim_{n\to\infty}\frac{3^n}{3^{n-1}} = 3$$	$$\lim_{n\to\infty}\frac{3^n-5^n}{4^n+3^n+2^n}$$
12.	$$\lim_{n\to\infty}\frac{1+2+2^2+\cdots+2^{n-1}}{1+3+3^2+\cdots+3^{n-1}}$$ $$1+q+q^2+\cdots+q^{n-1} = \frac{q^n-1}{q-1}$$	$$\lim_{n\to\infty}\frac{1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\cdots+\left(\frac{1}{2}\right)^{n-1}}{1+\frac{1}{3}+\left(\frac{1}{3}\right)^2+\cdots+\left(\frac{1}{3}\right)^{n-1}}$$

	$= \lim\limits_{n\to\infty} \dfrac{\dfrac{2^n-1}{2-1}}{\dfrac{3^n-1}{3-1}} = \lim\limits_{n\to\infty} \dfrac{2\cdot 2^n}{3^n} = \lim\limits_{n\to\infty} 2\cdot\left(\dfrac{2}{3}\right)^n = 0,$ $\dfrac{2}{3} \in (-1,1)$ atunci $\left(\dfrac{2}{3}\right)^n \to 0$	
13.	$\lim\limits_{n\to\infty}\big(ln(8n-1) - ln(n+8)\big) = \infty - \infty =$ $= \lim\limits_{n\to\infty} ln\dfrac{8n-1}{n+8} = \lim\limits_{n\to\infty} ln\dfrac{8n}{n} = ln8$	$\lim\limits_{n\to\infty}(ln(e^{2n}+4) - n)$
14.	$\lim\limits_{n\to\infty}\left(1+\dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{n}\right) =$ *utilizăm șirul remarcabil* $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{n}-\ln n\right) = c,$ $c = 0{,}57\ldots$ *constanta lui Euler* $= \lim\limits_{n\to\infty}\left(1+\dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{n}-\ln n+\ln n\right) =$ $= \lim\limits_{n\to\infty}(c+\ln n) = \infty$	$\lim\limits_{n\to\infty}\left(\dfrac{1}{n+1}+\dfrac{1}{n+2}+\cdots+\dfrac{1}{2n}\right)$
	Lema Stolz-Cesaro *Dacă șirurile* $(a_n),(b_n)$ *au proprietățile:* 1) (b_n)*are termeni nenuli și este crescător și nemărginit,* 2) $\lim\limits_{n\to\infty}\dfrac{a_{n+1}-a_n}{b_{n+1}-b_n} = a,$ *atunci* $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = a.$	
15.	$\lim\limits_{n\to\infty}\dfrac{1+\dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{n}}{n} = \dfrac{\infty}{\infty} = Stolz - Cesaro$ $= \lim\limits_{n\to\infty}\dfrac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\frac{1}{n+1}-\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)}{n+1-n}$ $= \lim\limits_{n\to\infty}\dfrac{1}{n+1} = 0$	$\lim\limits_{n\to\infty}\dfrac{1+\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}+\cdots+\dfrac{1}{\sqrt{n}}}{\sqrt{n}}$
	Criteriul raportului *Fie șirul* (a_n) *de numere strict pozitive* $a.$ î. $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = a.$ *Atunci*: 1)*dacă* $a<1 \Rightarrow \lim\limits_{n\to\infty} a_n = 0,$ 2)*dacă* $a>1 \Rightarrow \lim\limits_{n\to\infty} a_n = \infty$	

16.	$\lim\limits_{n\to\infty}\frac{5^n}{n!}=$ *utilizăm Criteriul raportului* $\lim\limits_{n\to\infty}\frac{\frac{5^{n+1}}{(n+1)!}}{\frac{5^n}{n!}}=\lim\limits_{n\to\infty}\frac{5}{n+1}=0<1\Rightarrow\lim\limits_{n\to\infty}\frac{5^n}{n!}=0$	$\lim\limits_{n\to\infty}\frac{3^n}{n!}$
	Criteriul radicalului.Cauchy-d'Alembert *Fie șirul* (a_n) *de numere strict pozitive.* $Dacă\ \lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=a, atunci\ \lim\limits_{n\to\infty}\sqrt[n]{a_n}=a.$	
17.	$\lim\limits_{n\to\infty}\sqrt[n]{n^2+n+1}$ *utilizăm Criteriul radicalului* $\lim\limits_{n\to\infty}\frac{(n+1)^2+(n+1)+1}{n^2+n+1}=\lim\limits_{n\to\infty}\frac{n^2}{n^2}=1\Rightarrow$ $\lim\limits_{n\to\infty}\sqrt[n]{n^2+n+1}=1$	$\lim\limits_{n\to\infty}\sqrt[n]{n!}$
	Criteriu de convergență $(a_n)mărginit, b_n\to 0\Rightarrow a_n\cdot b_n\to 0$	
18.	$\lim\limits_{n\to\infty}\frac{\sin n}{n}=0$ *utilizând Criteriul de convergență* $a_n=\sin n\in[-1,1]$ *și* $b_n=\frac{1}{n}\to 0$	$\lim\limits_{n\to\infty}\left(\frac{1}{2}\right)^n\cos n^2$
19.	$\lim\limits_{n\to\infty}\frac{sinx_n}{x_n}=1, x_n\in\mathbb{R}^*, x_n\to 0$ $\lim\limits_{n\to\infty}\frac{sin\frac{1}{n}}{\frac{1}{n}}=1$	$\lim\limits_{n\to\infty}\frac{sin\frac{n+1}{n^2}}{\frac{n+1}{n^2}}$
20.	$\lim\limits_{n\to\infty}\frac{tgx_n}{x_n}=1, x_n\in\mathbb{R}^*, x_n\to 0$ $\lim\limits_{n\to\infty}\frac{tg\frac{n+1}{n^2}}{\frac{n+1}{n^2}}=1$	$\lim\limits_{n\to\infty}\frac{tg\frac{1}{3^n}}{\frac{19}{3^n}}$

21.	$\lim\limits_{n\to\infty}\dfrac{arcsinx_n}{x_n}=1, x_n\in\mathbb{R}^*, x_n\to 0$ $\lim\limits_{n\to\infty}\dfrac{arcsin\dfrac{3}{n}}{\dfrac{3}{n}}=1$	$\lim\limits_{n\to\infty}\dfrac{arcsin\dfrac{\sqrt{2}}{n}}{\dfrac{1}{n}}$
22.	$\lim\limits_{n\to\infty}\dfrac{arctgx_n}{x_n}=1, x_n\in\mathbb{R}^*, x_n\to 0$ $\lim\limits_{n\to\infty}\dfrac{arctg\dfrac{\sqrt{2}}{n}}{\dfrac{1}{n}}=\lim\limits_{n\to\infty}\dfrac{arctg\dfrac{\sqrt{2}}{n}}{\sqrt{2}\dfrac{1}{n}}\sqrt{2}=\sqrt{2}$	$\lim\limits_{n\to\infty}\dfrac{arctg\dfrac{1}{2\sqrt{n}}}{\dfrac{1}{\sqrt{n}}}$
23.	$\lim\limits_{n\to\infty}\dfrac{\ln(1+x_n)}{x_n}=1, x_n\in\mathbb{R}^*, x_n\to 0, 1+x_n>0$ $\lim\limits_{n\to\infty}\dfrac{\ln(1+\dfrac{1}{\sqrt{n}})}{\dfrac{1}{\sqrt{n}}}=1$	$\lim\limits_{n\to\infty}\dfrac{\ln\left(\dfrac{n+1}{n}\right)}{\dfrac{1}{n}}$
24.	$\lim\limits_{n\to\infty}\dfrac{a^{x_n}-1}{x_n}=\ln a\,, x_n\in\mathbb{R}^*, x_n\to 0, a>0, a\neq 1$ $\lim\limits_{n\to\infty}n\left(2^{\frac{1}{n}}-1\right)=\ln 2$	$\lim\limits_{n\to\infty}\dfrac{\sqrt[n]{5}-1}{\dfrac{1}{n}}$
25.	$\lim\limits_{n\to\infty}\dfrac{(1+x_n)^r-1}{x_n}=r, x_n\in\mathbb{R}^*, x_n\to 0, r\in\mathbb{R}$ $\lim\limits_{n\to\infty}\dfrac{\left(\dfrac{n+3}{n}\right)^7-1}{\dfrac{3}{n}}=7$	$\lim\limits_{n\to\infty}n^2\left(\sqrt[21]{\dfrac{n^2+7}{n^2}}-1\right)$